Introduction to Modern Cryptography

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Finite Groups, Rings, and Fields
Lecture 2 Part b

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Review - Commutative Groups

Definition: A non-empty set G with a binary operation $+$ (addition) is called a commutative group if

1. $\forall a, b \in G, a + b \in G$ (closure under $+$).
2. $\forall a, b, c \in G$, $(a + b) + c = a + (b + c)$ (associativity).
3. $\forall a, b \in G$, $a + b = b + a$ (commutativity).
4. $\exists\, 0 \in G$ such that $\forall a \in G\, a + 0 = a$ (neutral element).
5. $\forall a \in G, \exists b \in G, a + b = 0$ (existence of inverse).

Note that $+$ and 0 are just symbols. In a multiplicative and/or non-commutative context, $\cdot$ and 1 are often used instead.

Sub-groups

- Let $(G, +)$ be a group. $(H, +)$ is called a sub-group of $(G, +)$ if it is a group, and $H \subset G$.
- Claim: Let $(G, +)$ be a finite group, and $H \subset G$. If H is closed under $+$, then $(H, +)$ is a sub-group of $(G, +)$.
- Question: What happens in the infinite case?

- Lagrange Theorem: If $(G, +)$ is a finite group and $(H, +)$ is a sub-group of it, then $|H|$ divides $|G|$.
- Remark: Lagrange theorem holds for non-commutative groups as well.

Order of Group Elements

- Let a^n denote $a + a + \ldots + a$ (n times).
- We say that a is of order n if $a^n = 0$, but for every $m < n$, $a^m \neq 0$.
- Euler theorem: In Z_m^*, the multiplicative group of Z_m^*, each element is of order at most $\phi(m)$.
- We will omit the operation from the group notation $(G, +)$ when it is obvious.

Cyclic Groups

- Claim: Let G be a group, and a an element of order n. The set $\langle a \rangle = \{0, a, \ldots, a^{n-1}\}$ is a subgroup of G.
- a is called the generator of $\langle a \rangle$.
- By Lagrange theorem, for every $a \in G$, the order of a divides $|G|$.
- Fermat's "little" theorem: For every $a \in \{1, \ldots, p-1\}$, $a^{p-1} \mod p = 1$ (why does this hold?).

- If G is generated by some a then G is called cyclic, and a is called a primitive element of G.
- Theorem: For any prime p, the multiplicative group Z_p^* is cyclic.
- Question: How many primitive elements does Z_p^* have? If we know one, say g, can we characterize the others?

Review (or maybe not) - Rings

Definition: A non-empty set R with two binary operation $+$ (addition) and $\cdot$ (multiplication) is called a commutative ring with identity if

1. $\forall a, b \in R$, $a + b, a \cdot b \in R$ (closure under $+, \cdot$).
2. $\forall a, b, c \in R$, $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of $+, \cdot$).
3. $\forall a, b \in R$, $a + b = b + a$, $a \cdot b = b \cdot a$ (commutativity of $+, \cdot$).
4. $\exists\, 0, 1 \in R$ such that $\forall a \in R$, $a + 0 = a \cdot 1 = a$ (neutral elements w.r.t. $+, \cdot$).
5. $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity of $+$ w.r.t. $\cdot$).
6. $\forall a \in R\, \exists b \in R, a + b = 0$ (existence of additive inverse).

Again, $+, \cdot$ and $0, 1$ are just symbols. Note that we did not require the existence of multiplicative inverses.

Fields

Definition: A non-empty set F with two binary operation $+$ (addition) and $\cdot$ (multiplication) is called a field if

1. $\forall a, b \in F,\ a + b, a \cdot b \in F$ (closure under $+, \cdot$).
2. $\forall a, b, c \in F,\ (a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of $+, \cdot$).
3. $\forall a, b \in F,\ a + b = b + a,\ a \cdot b = b \cdot a$ (commutativity of $+, \cdot$).
4. $\exists\, 0, 1 \in F$ such that $\forall a \in F\ a + 0 = a \cdot 1 = a$ (neutral elements w.r.t. $+, \cdot$).
5. $\forall a, b, c \in F,\ a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity of $+$ w.r.t. $\cdot$).
6. $\forall a \in F\, \exists b \in F, a + b = 0$ (existence of additive inverse).
7. $\forall a \neq 0 \in F\, \exists c \in F, a \cdot c = 1$ (existence of multiplicative inverse).

Fields – Recap

A field is a commutative ring with identity where each non-zero element has a multiplicative inverse:

$$\forall a \neq 0 \in F \, \exists c \in F, a \cdot c = 1.$$

The multiplicative inverse of a is also denoted a^{-1}.

Equivalently, $(F, +)$ is a commutative (additive) group, and $(F \setminus \{0\}, \cdot)$ is a commutative (multiplicative) group.

Univariate Polynomials over Fields

Let $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \ldots + a_1 x + a_0$ be a polynomial of degree n in one variable x over a field F (namely $a_n, a_{n-1}, \ldots, a_1, a_0 \in F$).

Theorem: The equation $f(x) = 0$ has at most n solutions in F.
(Such solution is called a root of $f(x)$.)

Remark: The theorem does not hold over rings with identity.
For example, in Z_{24}, the equation $6x = 0$ has six roots $(0, 4, 8, 12, 16, 20)$, not just one.

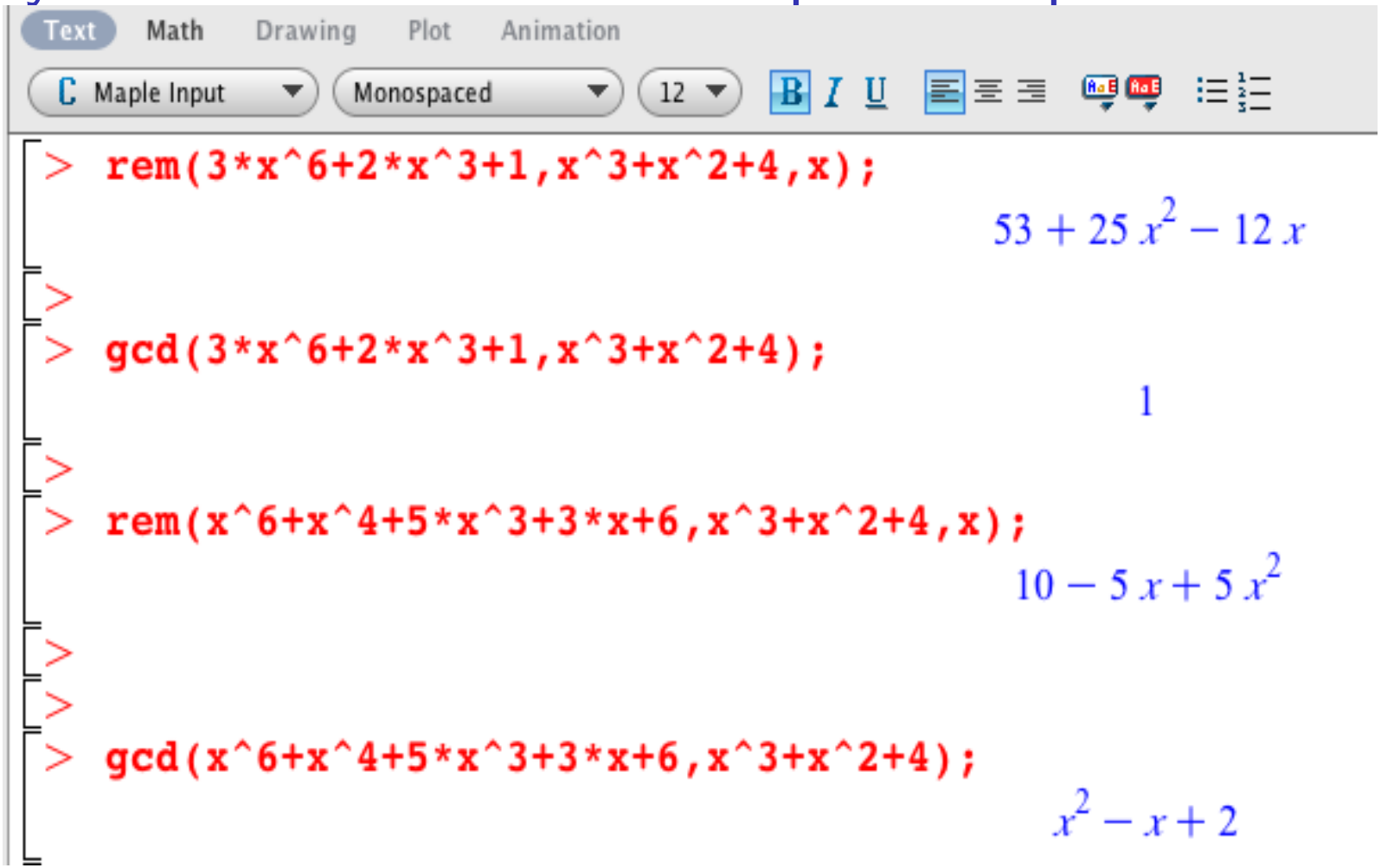
Polynomial Remainders

Let $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \ldots + a_1 x + a_0$
$g(x) = b_m \cdot x^m + b_{m-1} \cdot x^{m-1} + b_{m-2} \cdot x^{m-2} + \ldots + b_1 x + b_0$ be two polynomials in one variable x over a field F such that $m \leq n$.

Theorem: There is a unique polynomial $r(x)$ of degree smaller than m, and another unique polynomial, $h(x)$, both over F, such that $f(x) = h(x) \cdot g(x) + r(x)$.

The polynomial $r(x)$ is called the remainder of $f(x)$ modulo $g(x)$.

Polynomial Remainders: A Maple Example



The remainder is 0 (the zero polynomial) iff $g(x) \mid f(x)$. But if this is not the case, the remainder may be of degree $m-1$, while $f(x)$ and $g(x)$ may or may not be relatively prime.

Finite Fields

Definition: A field $(F, +, \cdot)$ is called a finite field if the set F is finite.

Example: As we already saw, Z_p denotes the set $\{0, 1, ..., p-1\}$, where we define $+$ and $\cdot$ as addition and multiplication modulo p, respectively.

It is not hard to prove that $(Z_p, +, \cdot)$ is a field iff p is a prime. (try this!)

It is also possible to show that for any prime, p, $(Z_p, +, \cdot)$ is the only finite field with p elements. This means that any finite field with that many elements is essentially $(Z_p, +, \cdot)$ (up to changing names).

Question: Are there any finite fields except $(Z_p, +, \cdot)$?

The Characteristic of Finite Fields

Let $(F, +, \cdot)$ be a finite field.

There must be a positive integer, n, such that $1 + 1 + \ldots + 1$ (n times) equals 0.

The mimimal such n is called the characteristic of F, char(F).

Theorem: For any finite field F, char(F) is a prime number.

Galois Fields $GF(p^k)$

Theorem: For every prime power p^k $(k = 1, 2, \ldots)$ there is a unique finite field with p^k elements (unique up to renaming). These fields are denoted by $GF(p^k)$. There are no finite fields with other cardinalities.

Évariste Galois (1811-1832)

(`http://www.wqsb.qc.ca/philemon/pmessier/mathematicians.htm`)

Remarks:

1. For $F = GF(p^k)$, $\text{char}(F) = p$.
2. $GF(p^k)$ and Z_{p^k} are not the same!

Polynomials over Finite Fields

Polynomial equations and factorizations over finite fields can be quite different than their rationals/reals counterparts.

Examples from a Maple session:

```
> factor(x^6-1);  # over the rationals/reals
```

$$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$$

```
> p:=7;
```

$$p := 7$$

```
> f := modp1(ConvertIn(x^6-1,x),p);
```

$$f := (x^6+6) \bmod 7$$

```
>
> modp1(Factors(f),p)[2]; # list of factors and their multiplicities over GF(7)
```

$$[[(x+5) \bmod 7, 1], [(x+6) \bmod 7, 1], [(x+2) \bmod 7, 1], [(x+1) \bmod 7, 1], [(x+4) \bmod 7, 1], [(x+3) \bmod 7, 1]]$$

```
> p:=2;
```

$$p := 2$$

```
> g := modp1(ConvertIn(x^6-1,x),p);
```

$$g := (x^6+1) \bmod 2$$

```
> modp1(Factors(g),p)[2]; # list of factors and their multiplicities over GF(2)
```

$$[[(x+1) \bmod 2, 2], [(x^2+x+1) \bmod 2, 2]]$$

Over $GF(7), x^6 - 1$ has six linear factors (btw, is this a coincidence?), while over $GF(2)$ the factorization is the same as over the rationals (given that $-1 = 1$).

Irreducible Polynomials

A polynomial is irreducible over $GF(p)$ if it does not factor in $GF(p)$. Otherwise, it is called reducible.

Maple example:

```
> p:=2;
```

$$p := 2$$

```
> f := modp1(ConvertIn(x^5+x^3+1,x),p);
```

$$f := \left(x^5 + x^3 + 1\right) \bmod 2$$

```
> modp1(Factors(f),p)[2]; # list of factors and their multiplicities over GF(2)
```

$$\left[\left[\left(x^5 + x^3 + 1\right) \bmod 2,\ 1\right]\right]$$

```
> p:=5;
```

$$p := 5$$

```
> g := modp1(ConvertIn(x^5+x^3+1,x),p);
```

$$g := \left(x^5 + x^3 + 1\right) \bmod 5$$

```
> modp1(Factors(g),p)[2]; # list of factors and their multiplicities over GF(5)
```

$$\left[\left[\left(x^2 + 2x + 3\right) \bmod 5,\ 1\right],\ \left[\left(x^3 + 3x^2 + 2x + 2\right) \bmod 5,\ 1\right]\right]$$

$x^5 + x^3 + 1$ is irreducible over $GF(2)$, but reducible over $GF(5)$.

Implementing $GF(p^k)$ Arithmetic

Theorem: Let $f(x)$ be an irreducible polynomial of degree k over $GF(p)$.

The arithmetic of the finite field $GF(p^k)$ can be realized by the set of polynomials over $GF(p)$ whose degree is at most $k-1$, where addition and multiplication are done modulo $f(x)$.

Comment: For every p, k there are irreducible polynomials of degree k over $GF(p)$. Furthermore, such polynomial can be found efficiently (random polynomial time in $\log p$ and k).

Example: Implementing $GF(2^5)$

By the theorem, the finite field $GF(2^5)$ can be realized as the set of degree 4 polynomials over Z_2, with addition and multiplication done modulo the irreducible polynomial $f(x) = x^5 + x^3 + 1$.

Remark: $f(x) = x^5 + x^3 + 1$ is not the only irreducible polynomial over Z_2. But it does not matter which (irreducible) one we take – they all give the same object, $GF(2^5)$.

The coefficients of polynomials over Z_2 are 0 or 1. So a degree $k-1$ polynomial can be written down by k bits. For example, with $k = 5$:

- $x^3 + x + 1$ is represented by $(0, 1, 0, 1, 1)$
- $x^4 + x^3 + x + 1$ is represented by $(1, 1, 0, 1, 1)$

Implementing Addition in $GF(p^k)$

In fields of characteristic 2, $1 + 1 = 0$, so addition corresponds to bit-wise XOR.

For example, $(x^3 + x + 1) + (x^4 + x^3 + x + 1)$ corresponds to

$$(0, 1, 0, 1, 1) \bigoplus (1, 1, 0, 1, 1),$$

which equals $(1, 0, 0, 0, 0)$, so

$(x^3 + x + 1) + (x^4 + x^3 + x + 1) = x^4$.

For fields of larger characteristic $p > 2$, the procedure is the same, only instead of XOR we do $\mod p$ addition.

Implementing Multiplication in $GF(p^k)$

Multiplication has two stages. First stage is polynomial multiplication, which results in a polynomial of degree (at most) $2k-2$. The second stage is computing the remainder of this polynomial modulo the defining, irreducible polynomial of degree k, $f(x)$, doing the computation mod p.

Maple example, in $GF(2^5)$, with $f(x) = x^5 + x^3 + 1$:

```
> p(x):=expand((x^3+x+1)*(x^4+x^3+x+1));
```

$$p(x) := x^7 + x^6 + 3\,x^4 + 2\,x^3 + x^5 + x^2 + 2\,x + 1$$

```
>
> rem(p(x),x^5+x^3+1,x) mod 2;
```

$$1 + x$$

So $(0,1,0,1,1) \cdot (1,1,0,1,1)$ equals $(0,0,0,1,1)$.

For small size finite field, a lookup table is the most efficient method for implementing multiplication.

Example: Implementing $GF(2^5)$ with Maple

Again, we take the irreducible polynomial $f(x) = x^5 + x^3 + 1$.

```
> G32:=GF(2,5,x^5+x^3+1);
```

$$G32 := Z_2[x] \,/\, \langle x^5 + x^3 + 1 \rangle$$

```
> a:=G32[ConvertIn](x); # slightly cumbersome notation
```

$$a := x \bmod 2$$

```
> b:=G32[`^`](a,8): # raising to power 8
> c:=G32[`^`](a,9): # colon after statement supresses
  printing
> G32[ConvertOut](b); # back to regular representation
```

$$x^4 + x^3 + x$$

```
> G32[ConvertOut](c);
```

$$x^4 + x^3 + x^2 + 1$$

```
> d:=G32[ConvertIn](x^3+x+1);
```

$$d := \left(x^3 + x + 1\right) \bmod 2$$

```
> e:=G32[`^`](d,8):
> G32[ConvertOut](e);
```

$$x^2 + x + 1$$

More Operations in $GF(p^k)$ with Maple

We continuing the previous slide, with the finite field $GF(2^5)$ represented using the irreducible polynomial $f(x) = x^5 + x^3 + 1$. We now check whether elements are primitive, namely if they are generators of the multiplicative group $GF^*(2^5)$.

```
> e:=G32[`^`](d,31):
> G32[ConvertOut](e);
                                   1
> G32[isPrimitiveElement](a);
                                 true
> G32[isPrimitiveElement](b);
                                 true
> G32[isPrimitiveElement](c);
                                 true
> G32[isPrimitiveElement](d);
                                 true
> G32[isPrimitiveElement](e);
                                 false
```